

1-COHOMOLOGY OF SIMPLICIAL AMALGAMS OF GROUPS

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ABSTRACT. We develop a cohomological method to classify amalgams of groups. We generalize this to simplicial amalgams in any concrete category. We compute the non-commutative 1-cohomology for several examples of amalgams defined over small simplices.
Keywords: complexes of groups amalgam cohomology Goldschmidt's lemma incidence geometries
MSC : 20F05 20E06.

Acknowledgements. Part of the work for this paper was completed during an RiP visit to the Mathematisches Forschungsinstitut Oberwolfach in Spring of 2011. The authors would like to express their gratitude for the wonderful research environment.

1. Introduction

Recognizing the completion G of an amalgam from the multiplication table of that amalgam can be viewed as playing a Sudoku game on the multiplication table of G . More generally, the aim of the game is to decide what G might look like: You are given a set of subgroups and their intersections and you need to decide what the largest group containing such a structure can be. This approach is very useful for example in the classification of finite simple groups. More precisely, induction and local analysis provides a set of subgroups of the minimal counterexample to the classification and then amalgam type results such as the Curtis-Tits and Phan theorems show that the group is known after all.

This leaves open the question of whether just the structure of the subgroups involved determines the group. Most approaches to this problem [4, 12, 14, 24] use induction together with a lemma by Goldschmidt [11, 13] that describes the isomorphism classes of amalgams of two groups in terms of double coset enumeration. The results presented here are much more general in that they address the classification of amalgams of any finite rank ≥ 2 and any number of groups.

In a recent work [5] we used Bass-Serre theory of graphs of groups to classify all possible amalgams of Curtis-Tits shape with a given diagram. This note describes a method for higher rank amalgams. In general an amalgam can be defined over an arbitrary partially ordered set. In this paper we shall only consider amalgams defined over the poset of faces of a simplicial complex. Goldschmidt's Lemma arises as the case where the simplicial complex is an edge on two vertices.

Our starting point is a connected simplicial complex $X = (V, \Sigma)$ and a fixed amalgam $\mathcal{G}_0 = \{G_\sigma, \psi_\tau^\sigma \mid \sigma \subseteq \tau, \sigma, \tau \in \Sigma\}$, where the connecting maps $\psi_\tau^\sigma: G_\tau \rightarrow G_\sigma$ are injective group homomorphisms whose image we shall denote $\overline{G}_{\sigma,\tau}$. We call such an amalgam a simplicial amalgam.

Note that if \mathcal{G}_0 is an amalgam, then it is also a complex of groups in the sense of Bass [2], Serre [22], and Haefliger [17]. The difference comes from the fact that in a complex of groups, for each chain of simplices $\sigma \subseteq \rho \subseteq \tau$ the diagram

$$\begin{array}{ccc} & & G_\sigma \\ & \nearrow & \uparrow \\ G_\rho & & \\ & \nwarrow & \uparrow \\ & & G_\tau \end{array}$$

is required to commute up to inner automorphism of G_σ , whereas in a simplicial amalgam, we insist that this inner automorphism be the identity. As a consequence, there are more complexes of groups than simplicial amalgams. Thus, the notion of isomorphism for complexes of groups is weaker than that of simplicial amalgams.

Our aim is to classify amalgams of type \mathcal{G}_0 , where we define an amalgam of type \mathcal{G}_0 to be an amalgam whose groups G_σ and $\overline{G}_{\sigma,\tau}$ are those of \mathcal{G}_0 (For precise definitions see Section 2). Thus the classification reduces to classifying the collections of connecting maps up to isomorphism of the resulting amalgam. To this end we first create a collection of automorphism groups $\mathcal{A}_0 = \{A_\sigma, \alpha_\tau^\sigma \mid \sigma \subseteq \tau \in \Sigma\}$, where

$$A_\sigma = \{g \in \text{Aut}(G_\sigma) \mid g(\overline{G}_{\sigma,\tau}) = \overline{G}_{\sigma,\tau} \text{ for all } \tau \text{ with } \sigma \subseteq \tau \in \Sigma\}$$

and, for each pair (σ, τ) with $\sigma \subseteq \tau$, we have a connecting (“restriction”) map $\alpha_\tau^\sigma : A_\sigma \rightarrow A_\tau$ given by $\text{ad}(\psi_\tau^\sigma)(f) = (\psi_\tau^\sigma)^{-1} \circ f \circ \psi_\tau^\sigma$. For the rest of the paper we will abusively denote by $(\psi_\tau^\sigma)^{-1}$ (respectively $(\varphi_\tau^\sigma)^{-1}$), the inverse of the isomorphism $\psi_\tau^\sigma : G_\tau \rightarrow \overline{G}_{\sigma,\tau}$ (respectively $\varphi_\tau^\sigma : G_\tau \rightarrow \overline{G}_{\sigma,\tau}$). We view \mathcal{A}_0 as a coefficient system on the simplicial complex X .

In Section 3, we define a non-commutative first cohomology *set* $H^1(X, \mathcal{A}_0)$ on X with coefficients in \mathcal{A}_0 and in Section 4 we use this to prove the following result.

Theorem 1. *If X is a non-empty connected simplicial complex, then the isomorphism classes of amalgams of type \mathcal{G}_0 are parametrized by $H^1(X, \mathcal{A}_0)$.*

The correspondence between 1-cocycles and amalgams of type \mathcal{G}_0 is constructive, as is the correspondence between 1-coboundaries and isomorphisms between such amalgams.

In Section 6 we show that there is a result completely analogous to Theorem 1 in the much more general setting of simplicial amalgams in any concrete category.

Theorem 2. *Let ${}^0\mathcal{G}$ be a simplicial amalgam over X in a concrete category \mathbf{C} and let ${}^0\mathcal{A}$ be the associated coefficient system. Then, the isomorphism classes of amalgams of type ${}^0\mathcal{G}$ are parametrized by $H^1(X, {}^0\mathcal{A})$.*

For definitions and notation see Section 6. We finish the paper by illustrating the use of Theorem 2 with some examples.

2. Amalgams and complexes of groups

Definition 2.1. We define a *simplicial complex* to be a pair $X = (V, \Sigma)$ where V is a set of vertices and $\Sigma \subseteq \mathcal{P}(V)$ is a collection of finite subsets of V with the property that $\{v\} \in \Sigma$

for every $v \in V$ and if $\tau \in \Sigma$, then any subset $\sigma \subseteq \tau$ also belongs to Σ . An element $\sigma \in \Sigma$ is called a simplex of rank $|\sigma| - 1$. The boundary $\partial\tau$ of a simplex τ consists of all simplices of rank $|\tau| - 2$ contained in τ .

From now on we fix a particular connected simplicial complex $X = (V, \Sigma)$, with $V = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}_{\geq 1}$. Given a simplex $\tau = \{i_1, i_2, \dots, i_k\}$ with $i_1 < i_2 < \dots < i_k$, we have $\partial\tau = \{\tau_1, \dots, \tau_k\}$, where $\tau_j = \tau - \{i_j\}$ for all $j = 1, \dots, k$. The natural ordering of V now induces an ordering on $\partial\tau$ in which $\tau_j < \tau_l$, whenever $j < l$. We shall write $\bar{\tau} = \tau_1$.

Definition 2.2. A *simplicial amalgam* over the complex $X = (V, \Sigma)$ is a collection $\mathcal{G} = \{G_\sigma, \varphi_\tau^\sigma \mid \sigma \subseteq \tau, \sigma, \tau \in \Sigma\}$, where each G_σ is a group and, for each pair (σ, τ) such that $\sigma \subseteq \tau$ we have a monomorphism $\varphi_\tau^\sigma: G_\tau \hookrightarrow G_\sigma$, called an *inclusion map* such that, whenever $\sigma \subseteq \rho \subseteq \tau$, we have $\varphi_\rho^\sigma \circ \varphi_\tau^\rho = \varphi_\tau^\sigma$. For simplicity we shall write $\bar{G}_{\sigma, \tau} = \varphi_\tau^\sigma(G_\tau) \leq G_\sigma$. We shall use the shorthand notation $\mathcal{G} = \{G_\bullet, \varphi_\bullet^\bullet\}$.

A *completion* of \mathcal{G} is a group G together with a collection $\phi = \{\phi_\sigma \mid \sigma \in \Sigma\}$ of homomorphisms $\phi_\sigma: G_\sigma \rightarrow G$, such that whenever $\sigma \subseteq \tau$, we have $\phi_\sigma \circ \varphi_\tau^\sigma = \phi_\tau$. The amalgam \mathcal{G} is *non-collapsing* if it has a non-trivial completion. A completion $(\hat{G}, \hat{\phi})$ is called *universal* if for any completion (G, ϕ) there is a (necessarily unique) surjective group homomorphism $\pi: \hat{G} \rightarrow G$ such that $\phi = \pi \circ \hat{\phi}$.

Definition 2.3. We define a *homomorphism* between the amalgams $\mathcal{G}^{(1)} = \{G_\bullet^{(1)}, {}^{(1)}\varphi_\bullet^\bullet\}$ and $\mathcal{G}^{(2)} = \{G_\bullet^{(2)}, {}^{(2)}\varphi_\bullet^\bullet\}$ to be a map $\phi = \{\phi_\sigma \mid \sigma \in \Sigma\}$ where $\phi_\sigma: G_\sigma^{(1)} \rightarrow G_\sigma^{(2)}$ are group homomorphisms such that

$$(2.1) \quad \phi_\sigma \circ {}^{(1)}\varphi_\tau^\sigma = {}^{(2)}\varphi_\tau^\sigma \circ \phi_\tau.$$

We call ϕ an *isomorphism* of amalgams if ϕ_σ is bijective for all $\sigma \in \Sigma$.

Definition 2.4. Adopting the notation from Definition 2.3, suppose $\tilde{X} = (\tilde{V}, \tilde{\Sigma})$ is a simplicial complex such that $V \subseteq \tilde{V}$ and $\Sigma \subseteq \tilde{\Sigma}$. Given a simplicial amalgam $(\mathcal{G}_\bullet, \varphi_\bullet^\bullet)$ over X , we define a simplicial amalgam $(\tilde{\mathcal{G}}, \tilde{\varphi}_\bullet^\bullet)$ over \tilde{X} as follows.

$$\tilde{G}_\sigma = \begin{cases} G_\sigma & \text{if } \sigma \in \Sigma \\ \{1\} & \text{else} \end{cases} \quad \text{and} \quad \tilde{\varphi}_\tau^\sigma = \begin{cases} \varphi_\tau^\sigma & \text{if } \sigma, \tau \in \Sigma \\ \tilde{G}_\tau = \{1\} \hookrightarrow \tilde{G}_\sigma & \text{else.} \end{cases}$$

Now if $\phi = \{\phi_\sigma \mid \sigma \in \Sigma\}: \mathcal{G}^{(1)} \rightarrow \mathcal{G}^{(2)}$ is a homomorphism, then we define $\tilde{\phi} = \{\tilde{\phi}_\sigma \mid \sigma \in \tilde{\Sigma}\}: \tilde{\mathcal{G}}^{(1)} \rightarrow \tilde{\mathcal{G}}^{(2)}$ as follows.

$$\tilde{\phi}_\sigma = \begin{cases} \phi_\sigma & \text{if } \sigma \in \Sigma \\ \text{id}: \tilde{G}_\sigma^{(1)} = \{1\} \rightarrow \{1\} = \tilde{G}_\sigma^{(2)} & \text{else.} \end{cases}$$

Lemma 2.5. *With the notation of Definition 2.4 the assignment $\mathcal{G} \mapsto \tilde{\mathcal{G}}$ and $\phi \mapsto \tilde{\phi}$ is an embedding of the category of simplicial amalgams over X into the category of simplicial amalgams over \tilde{X} . In particular, $\phi: \mathcal{G}^{(1)} \xrightarrow{\cong} \mathcal{G}^{(2)}$ if and only if $\tilde{\phi}: \tilde{\mathcal{G}}^{(1)} \xrightarrow{\cong} \tilde{\mathcal{G}}^{(2)}$. Moreover, completions are preserved. That is (G, ϕ) is a completion of $(\mathcal{G}_\bullet, \varphi_\bullet^\bullet)$ if and only if $(G, \tilde{\phi})$ (defined in the obvious way) is a completion of $(\tilde{\mathcal{G}}_\bullet, \tilde{\varphi}_\bullet^\bullet)$.*

Proof This is a completely straightforward exercise. \square

Lemma 2.5 allows us to replace X by a simplicial complex of rank at least 2, if necessary. It also allows us to assume that all subsets of V of cardinality $\leq k$ are simplices in Σ . For the rest of the paper $\mathcal{G}_0 = \{G_\bullet, \psi_\bullet\}$ will be a fixed amalgam over $X = (V, \Sigma)$.

Definition 2.6. Let $\mathcal{G}_0 = \{G_\bullet, \psi_\bullet\}$ be an amalgam over $X = (V, \Sigma)$. An amalgam of type \mathcal{G}_0 is an amalgam $\mathcal{G} = \{G_\bullet, \varphi_\bullet\}$, where, for each $\sigma \in \Sigma$, the group G_σ is that of \mathcal{G}_0 and in which $\varphi_\tau^\sigma(G_\sigma) = \overline{G}_{\sigma, \tau} = \psi_\tau^\sigma(G_\sigma)$ for each pair (σ, τ) with $\sigma \subseteq \tau \in \Sigma$.

The aim of this note is to describe the isomorphism classes of all amalgams that have the same type as \mathcal{G}_0 . Note that the classification of amalgams of type \mathcal{G}_0 essentially comes down to classifying all collections of connecting maps $\{\varphi_\tau^\sigma \mid \sigma \subseteq \tau \in \Sigma\}$. Since we classify the amalgams up to isomorphism, some of the φ_τ^σ one can specify in advance.

Definition 2.7. We call an amalgam $\mathcal{G} = \{G_\sigma, \varphi_\tau^\sigma\}$ of type \mathcal{G}_0 *normalized* if for any simplex τ we have $\varphi_\tau^\tau = \psi_\tau^\tau$, where τ is the least maximal face in $\partial\tau$.

Lemma 2.8. Suppose that \mathcal{G} is normalized as in Definition 2.7. Suppose that $\sigma \subseteq \tau$ have the same largest element i . Then, $\varphi_\tau^\sigma = \psi_\tau^\sigma$.

Proof Note that, by induction on $|\sigma|$ and $|\tau|$ we have $\varphi_\sigma^i = \psi_\sigma^i$ and $\varphi_\tau^i = \psi_\tau^i$. Indeed $|\sigma| - 1$ applications of the map $\rho \mapsto \bar{\rho}$ leave the simplex $\{i\}$. Now since $\psi_\tau^i = \psi_\sigma^i \circ \varphi_\tau^\sigma = \psi_\sigma^i \circ \psi_\tau^\sigma$ and all maps are injective, the claim follows. \square

Proposition 2.9. Every amalgam of type \mathcal{G}_0 is isomorphic to a normalized amalgam.

Proof Let $\mathcal{G}^{(1)} = \{G_\bullet, \varphi_\bullet\}$ be an arbitrary amalgam of type \mathcal{G}_0 . We will construct a normalized amalgam $\mathcal{G}^{(2)} = \{G_\bullet, \varsigma_\bullet\}$ along with an isomorphism $\phi: \mathcal{G}^{(1)} \rightarrow \mathcal{G}^{(2)}$. We will define ς_τ^σ and ϕ_τ by induction on the rank of τ . To start the induction let $\phi_\tau = \text{id}$ for all simplices τ of rank 0. Assume that all ς_τ^σ and ϕ_τ have been defined for τ of rank at most $s \geq 0$. Now let τ be a simplex of rank $s + 1$. Define

$$\phi_\tau = (\psi_\tau^\tau)^{-1} \phi_\tau \varphi_\tau^\tau,$$

where τ is the least maximal face in $\partial\tau$.

Next, for each σ define ς_τ^σ via Equation (2.1) to be $\varsigma_\tau^\sigma = \phi_\sigma \circ \varphi_\tau^\sigma \circ \phi_\tau^{-1}$. A direct verification shows that, for any triple $\sigma \subseteq \rho \subseteq \tau$, we have $\varsigma_\rho^\sigma \circ \varsigma_\tau^\rho = \varsigma_\tau^\sigma$.

It now follows by definition that $\mathcal{G}^{(2)}$ is normalized and that ϕ is an isomorphism. \square

Proposition 2.9 says that we only need to classify normalized amalgams up to isomorphism.

Example 2.10. Consider a group G acting flag-transitively on a geometry $\Gamma = (\mathcal{O}, I, \tau, *)$, where \mathcal{O} denotes the set of objects, I denotes the set of types, $\text{typ}: \mathcal{O} \rightarrow I$ is a type map and $*$ is a symmetric reflexive relation on \mathcal{O} called the incidence relation. In the terminology of Buekenhout we shall assume that Γ is connected, transversal, and residually connected. Let X be the simplicial complex in which $V = I$ and $\Sigma = \mathcal{P}(V) - \{\emptyset\}$. Fix a chamber (maximal flag) $F = (o_i)_{i \in I}$, and for each non-empty subset $\sigma \subseteq I$, let $F_\sigma = (o_j)_{j \in \sigma}$. We

now define an amalgam $\mathcal{G}_0 = \{G_\bullet, \psi_\bullet\}$ over X setting $G_\sigma = \text{Stab}_G(F_\sigma)$, for each $\sigma \in \Sigma$ and letting $\psi_\tau^\sigma: G_\tau \rightarrow G_\sigma$ be the inclusion map of subgroups of G whenever $\sigma \subseteq \tau$. The group G_σ is called the standard parabolic subgroup of type σ . A result due to Soulé, Tits, and Pasini now says that G is the universal completion of \mathcal{G}_0 if and only if the complex whose simplices are the flags of Γ is simply-connected.

3. Coefficient systems and 1-cohomology

Let $X = (V, \Sigma)$ be a simplicial complex. For any $k \in \mathbb{N}$, let Σ_k be the set of all simplices of rank k and, for $l \in \mathbb{N}$, let $\Sigma_{\leq l} = \bigcup_{0 \leq k \leq l} \Sigma_k$.

Definition 3.1. A *coefficient system* on the simplicial complex $X = (V, \Sigma)$ is a collection

$$\mathcal{A} = \{A_\sigma, \alpha_\tau^\sigma \mid \sigma \subseteq \tau \text{ with } \sigma, \tau \in \Sigma\},$$

where A_σ is a group and $\alpha_\tau^\sigma: A_\sigma \rightarrow A_\tau$ is a group homomorphism such that whenever $\sigma \subseteq \rho \subseteq \tau$, we have $\alpha_\tau^\sigma = \alpha_\tau^\rho \circ \alpha_\rho^\sigma$. As for amalgams, we shall use the shorthand notation $\mathcal{A} = \{A_\bullet, \alpha_\bullet\}$.

Definition 3.2. We define a *homomorphism* between coefficient systems $\mathcal{A}^{(2)} = \{A_\bullet^{(2)}, \alpha_\bullet^{(2)}\}$ and $\mathcal{A}^{(1)} = \{A_\bullet^{(1)}, \alpha_\bullet^{(1)}\}$ to be a map $\chi = \{\chi_\sigma \mid \sigma \in \Sigma\}$ where $\chi_\sigma: A_\sigma^{(2)} \rightarrow A_\sigma^{(1)}$ are group homomorphisms such that

$$(3.1) \quad \chi_\tau \circ {}^{(2)}\alpha_\tau^\sigma = {}^{(1)}\alpha_\tau^\sigma \circ \chi_\sigma.$$

We call χ an *isomorphism* of coefficient systems if χ_σ is bijective for all $\sigma \in \Sigma$.

Definition 3.3. Adopting the notation from Definition 3.2, suppose $\tilde{X} = (\tilde{V}, \tilde{\Sigma})$ is a simplicial complex such that $V \subseteq \tilde{V}$ and $\Sigma \subseteq \tilde{\Sigma}$. Given a coefficient system $(\mathcal{A}_\bullet, \alpha_\bullet)$ over X , we define a coefficient system $(\tilde{\mathcal{A}}_\bullet, \tilde{\alpha}_\bullet)$ over \tilde{X} as follows.

$$\tilde{A}_\sigma = \begin{cases} A_\sigma & \text{if } \sigma \in \Sigma \\ \{1\} & \text{else} \end{cases} \quad \text{and} \quad \tilde{\alpha}_\tau^\sigma = \begin{cases} \alpha_\tau^\sigma & \text{if } \sigma, \tau \in \Sigma \\ \tilde{A}_\sigma \rightarrow \tilde{A}_\tau = \{1\} & \text{else.} \end{cases}$$

Now if $\phi = \{\phi_\sigma \mid \sigma \in \Sigma\}: \mathcal{A}^{(2)} \rightarrow \mathcal{A}^{(1)}$ is a homomorphism, then we define $\tilde{\phi} = \{\tilde{\phi}_\sigma \mid \sigma \in \tilde{\Sigma}\}: \tilde{\mathcal{A}}^{(2)} \rightarrow \tilde{\mathcal{A}}^{(1)}$ as follows.

$$\tilde{\phi}_\sigma = \begin{cases} \phi_\sigma & \text{if } \sigma \in \Sigma \\ \text{id}: \tilde{A}_\sigma^{(2)} = \{1\} \rightarrow \{1\} = \tilde{A}_\sigma^{(1)} & \text{else.} \end{cases}$$

Lemma 3.4. *With the notation of Definition 3.3 the assignment $\mathcal{A} \mapsto \tilde{\mathcal{A}}$ and $\phi \mapsto \tilde{\phi}$ is an embedding of the category of simplicial amalgams over X into the category of coefficient systems over \tilde{X} . In particular, $\phi: \mathcal{A}^{(2)} \xrightarrow{\cong} \mathcal{A}^{(1)}$ if and only if $\tilde{\phi}: \tilde{\mathcal{A}}^{(2)} \xrightarrow{\cong} \tilde{\mathcal{A}}^{(1)}$.*

Proof This is a completely straightforward exercise. \square

Lemma 3.4 allows us to replace X by a simplicial complex of rank at least 2, if necessary. It also allows us to assume that all subsets of V of cardinality $\leq k$ are simplices in Σ . Given a coefficient system $\mathcal{A} = \{A_\bullet, \alpha_\bullet\}$ on X , we define a cochain complex of pointed sets

$$\mathcal{C}: \mathcal{C}^0 \xrightarrow{d_0} \mathcal{C}^1 \xrightarrow{d_1} \mathcal{C}^2$$

where $(\mathcal{C}^i, \text{id}_i) = \prod_{\sigma \in \Sigma_i} (A_\sigma, \text{id}_{A_\sigma})$ as a product of pointed sets,

$$d_0((a_1, \dots, a_n)) = (b_{ij} \mid \{i, j\} \in \Sigma, i < j), \text{ where } b_{ij} = \alpha_{ij}^j(a_j^{-1})\alpha_{ij}^i(a_i).$$

and

$$d_1((a_{ij} \mid \{i, j\} \in \Sigma, i < j)) = (b_{ijk} \mid \{i, j, k\} \in \Sigma, i < j < k)$$

where

$$b_{ijk} = \alpha_{ijk}^{jk}(a_{jk}^{-1})\alpha_{ijk}^{ik}(a_{ik})\alpha_{ijk}^{ij}(a_{ij}^{-1}).$$

Note that the maps d_i are not necessarily group homomorphisms, although they can be, for instance when the A_σ 's are abelian groups. Therefore this is not a chain complex of groups, but merely a chain complex of pointed sets, where the pointing identifies the identity as a base point in each group A_σ . It is easy to see that the maps d_i preserve the base point.

Lemma 3.5. *We have $d_1 \circ d_0(\mathcal{C}^0) = (\text{id}_\tau)_{\tau \in \Sigma_2} = \text{id}_2$.*

Proof It suffices to prove that for any σ of rank 0 and any τ of rank 2, the composition

$$A_\sigma \hookrightarrow \mathcal{C}^0 \xrightarrow{d_0} \mathcal{C}^1 \xrightarrow{d_1} \mathcal{C}^2 \rightarrow A_\tau$$

sends A_σ to id_{A_τ} . Since the diagram of α 's is commutative, this is a computation in ordinary cohomology theory with α_τ^σ applied. \square

Definition 3.6. For $i = 0, 1$, the set of i -cocycles of \mathcal{C} is $Z^i(X, \mathcal{A}) = \{z \in \mathcal{C}^i \mid d_i(z) = \text{id}_{i+1}\}$.

We now define a right action of \mathcal{C}^0 on \mathcal{C}^1 as follows:

$$(3.2) \quad (b_{ij} \mid 1 \leq i < j \leq n)^{(a_k \mid 1 \leq k \leq n)} = (\alpha_{ij}^j(a_j^{-1}) \cdot b_{ij} \cdot \alpha_{ij}^i(a_i) \mid 1 \leq i < j \leq n),$$

for any $(b_{ij} \mid 1 \leq i < j \leq n) \in \mathcal{C}^1$ and $(a_k \mid 1 \leq k \leq n) \in \mathcal{C}^0$.

Lemma 3.7. *Let $a, b \in \mathcal{C}^0$. Then, we have*

- (a) $d_0(a) = \text{id}_1^a$, and
- (b) $(d_0(a))^b = d_0(ab)$.

Proof Part (a) is immediate from the definition, and part (b) follows from part (a) together with the fact that \mathcal{C}^0 acts on \mathcal{C}^1 from the right. \square

Lemma 3.8. *The action of \mathcal{C}^0 on \mathcal{C}^1 preserves $Z^1(X, \mathcal{A})$.*

Proof Let $a = (a_1, \dots, a_n) \in \mathcal{C}_0$ and let $z = (z_{ij} \mid 1 \leq i < j \leq n) \in Z^1(X, \mathcal{A})$. Then $z^a = y = (y_{ij} = \alpha_{ij}^j(a_j^{-1}) \cdot z_{ij} \cdot \alpha_{ij}^i(a_i) \mid 1 \leq i < j \leq n)$. The projection of the co-boundary $d_1(z^a)$ on A_τ for some simplex $\tau = \{i, j, k\}$ with $i < j < k$ equals:

$$\alpha_{ijk}^{jk}(y_{jk}^{-1})\alpha_{ijk}^{ik}(y_{ik})\alpha_{ijk}^{ij}(y_{ij}^{-1})$$

Using the definition of y we get

$$\begin{aligned} \alpha_{ijk}^{jk}(\alpha_{jk}^j(a_j^{-1}) \cdot z_{jk}^{-1} \cdot \alpha_{jk}^k(a_k)) &\cdot \alpha_{ijk}^{ik}(\alpha_{ik}^k(a_k^{-1}) \cdot z_{ik} \cdot \alpha_{ik}^i(a_i)) \\ &\cdot \alpha_{ijk}^{ij}(\alpha_{ij}^i(a_i^{-1}) \cdot z_{ij}^{-1} \cdot \alpha_{ij}^j(a_j)) \end{aligned}$$

We use the composition property of the α -homomorphisms and some cancellations to get:

$$\alpha_{ijk}^j(a_j^{-1}) \cdot \alpha_{ijk}^{jk}(z_{jk}^{-1}) \cdot \alpha_{ijk}^{ik}(z_{ik}) \cdot \alpha_{ijk}^{ij}(z_{ij}^{-1}) \cdot \alpha_{ijk}^j(a_j),$$

and this equals $\text{id}_{A_{ijk}}$ since z is a cocycle. \square

Definition 3.9. The zero cohomology set is $H^0(X, \mathcal{A}) = Z^0(X, \mathcal{A})$. The orbits in $Z^1(X, \mathcal{A})$ under the action defined above are called 1-cohomology classes. The first cohomology set is the collection of \mathcal{C}_0 orbits of 1-cocycles. We write

$$H^1(X, \mathcal{A}) = Z^1(X, \mathcal{A})^{\mathcal{C}^0}.$$

Moreover, for each $z \in Z^i(X, \mathcal{A})$ we denote its cohomology class as $[z]$. Note that for $z \in Z^0(X, \mathcal{A})$ we have $[z] = \{z\}$.

Corollary 3.10. *The zero cohomology set $H^0(X, \mathcal{A})$ is a group.*

Proof This follows from Lemma 3.7 since $d_0(ab) = ((\text{id}_1)^a)^b = \text{id}_1^b = \text{id}_1$. \square

3.1. A cohomology exact sequence

Definition 3.11. A normal subsystem of \mathcal{A} is a collection $\{N_\sigma \mid \sigma \in \Sigma\}$ of normal subgroups $N_\sigma \triangleleft A_\sigma$ such that $\mathcal{N} = \{N_\bullet, \alpha_\bullet\}$ is a coefficient system on X . We shall denote this as $\mathcal{N} \triangleleft \mathcal{A}$. The normal sub coefficient system naturally gives rise to a quotient coefficient system by setting $\mathcal{A}/\mathcal{N} = \{A_\bullet/N_\bullet, \alpha_\bullet\}$. Note that for $i = 0, 1$, $\mathcal{C}^i(\mathcal{N}) \triangleleft \mathcal{C}^i(\mathcal{A})$ and therefore, $\mathcal{C}^i(\mathcal{A}/\mathcal{N}) \cong \mathcal{C}^i(\mathcal{A})/\mathcal{C}^i(\mathcal{N})$.

Theorem 3.12. *The group $H^0(X, \mathcal{A}/\mathcal{N})$ acts on $H^1(X, \mathcal{N})$ in a natural way. The orbits of this action are precisely the fibers of the map $H^1(X, \mathcal{N}) \xrightarrow{i_1} H^1(X, \mathcal{A})$, which takes each $\mathcal{C}^0(\mathcal{N})$ orbit on $Z^1(X, \mathcal{N})$ to the unique \mathcal{C}^0 orbit on $Z^1(X, \mathcal{A})$ that contains it.*

Proof Given $\bar{a} = (a_\sigma N_\sigma)_{\sigma \in \Sigma_0} \in H^0(X, \mathcal{A}/\mathcal{N})$ and $n = (n_\tau)_{\tau \in \Sigma_1} \in Z^1(X, \mathcal{N})$, let $a = (a_\sigma)_{\sigma \in \Sigma_0} \in \mathcal{C}^0(\mathcal{A})$. We define

$$[n]^{\bar{a}} = [n^a].$$

We claim that this is well-defined. Indeed, let $m \in \mathcal{C}^0(\mathcal{N})$ and $a \in \mathcal{C}^0(\mathcal{A})$ and let $m' \in \mathcal{C}^0(\mathcal{N})$ be such that $am = m'a$. Then, $n^{am} = n^{m'a} = (n^{m'})^a$ and so $[n^{am}] = [(n^{m'})^a] = [n^a]$. Suppose now that moreover, $\bar{a} \in Z^0(X, \mathcal{A}/\mathcal{N})$. Then, $d_0(a) \in \mathcal{C}^1(\mathcal{N})$. This means that for each $\{i, j\} \in \Sigma_1$ with $i < j$, we have $m_{ij} = (d_0(a))_{ij} = \alpha_{ij}^j(a_j^{-1})\alpha_{ij}^i(a_i) \in N_{ij}$. Hence, if $n = (n_{ij})_{\{i,j\} \in \Sigma_1}$, then,

$$(3.3) \quad n^a = (\alpha_{ij}^j(a_j^{-1}) n_{ij} \alpha_{ij}^i(a_i))_{\{i,j\} \in \Sigma_1} = (\alpha_{ij}^j(a_j^{-1}) n_{ij} \alpha_{ij}^j(a_j) m_{ij})_{\{i,j\} \in \Sigma_1}$$

and this belongs to $\mathcal{C}^1(\mathcal{N})$ since $N_{ij} \triangleleft A_{ij}$. This concludes the proof of our claim.

By definition $i_1([n^a])$ and $i_1([n])$ are in the same cohomology class of $H^1(X, \mathcal{A})$. Conversely, suppose that $[n]$ and $[n']$ are in $H^1(X, \mathcal{N})$ such that $i_1([n]) = i_1([n'])$. Then there is some $a \in \mathcal{C}^0(X, \mathcal{A})$ with $n' = n^a$. We claim that $d_0(a) \in \mathcal{C}^1(\mathcal{N})$ so that $[n]$ and $[n']$ are in the same $H^0(X, \mathcal{A}/\mathcal{N})$ -orbit. Indeed define $d_0(a) = (m_{ij})_{\{i,j\} \in \Sigma_1}$. Then Equation (3.3) still holds for $[n'] = [n^a]$. Since n' and n belong to $\mathcal{C}^1(\mathcal{N})$ and $\mathcal{C}^1(\mathcal{N}) \triangleleft \mathcal{C}^1(\mathcal{A})$, also $m \in \mathcal{C}^1(\mathcal{N})$. \square

Theorem 3.13. *For any $\mathcal{N} \triangleleft \mathcal{A}$ there is a natural exact sequence of pointed sets*

$$0 \rightarrow H^0(X, \mathcal{N}) \xrightarrow{i_0} H^0(X, \mathcal{A}) \xrightarrow{\kappa_0} H^0(X, \mathcal{A}/\mathcal{N}) \xrightarrow{\delta^*} H^1(X, \mathcal{N}) \xrightarrow{i_1} H^1(X, \mathcal{A}) \xrightarrow{\kappa_1} H^1(X, \mathcal{A}/\mathcal{N})$$

Proof For $j = 0, 1$, the map i_j is given by the inclusion maps $N_\sigma \hookrightarrow A_\sigma$ and the map κ_j is given by the canonical homomorphism $A_\sigma \rightarrow A_\sigma/N_\sigma$ for any $\sigma \in \Sigma$. The map δ^* is defined as

$$\delta^*((a_\sigma N_\sigma)_{\sigma \in \Sigma_0}) = d_0((a_\sigma)_{\sigma \in \Sigma_0}).$$

That this is well-defined can be seen as follows. Since the α_\bullet are group homomorphisms that preserve \mathcal{N} , it follows that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}^0(\mathcal{N}) & \longrightarrow & \mathcal{C}^0(\mathcal{A}) & \longrightarrow & \mathcal{C}^0(\mathcal{A}/\mathcal{N}) \longrightarrow 0 \\ & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\ 0 & \longrightarrow & \mathcal{C}^1(\mathcal{N}) & \longrightarrow & \mathcal{C}^1(\mathcal{A}) & \longrightarrow & \mathcal{C}^1(\mathcal{A}/\mathcal{N}) \longrightarrow 0 \\ & & \downarrow d_1 & & \downarrow d_1 & & \downarrow d_1 \\ 0 & \longrightarrow & \mathcal{C}^2(\mathcal{N}) & \longrightarrow & \mathcal{C}^2(\mathcal{A}) & \longrightarrow & \mathcal{C}^2(\mathcal{A}/\mathcal{N}) \longrightarrow 0 \end{array}$$

Note that the d_0 's and d_1 's are merely maps of pointed sets. Note that $d_1 \circ d_0 = 0$ and that the rows in the diagram are exact sequences of group homomorphisms. A pointed-set version of the Snake Lemma shows that δ^* is well-defined and that the sequence is exact up to $H^1(X, \mathcal{N})$. Exactness at $H^1(X, \mathcal{N})$ follows from Theorem 3.12. It is easy to see that $\text{im}(i_1) \leq \ker(\kappa_1)$. Let $[a] \in \ker(\kappa_1)$. Since $a \in Z^1(X, \mathcal{A})$, we have $d_1(a) = \text{id}_2$, and since $[a] \in \ker(\kappa_1)$ there exists some $b \in \mathcal{C}^0(\mathcal{A})$ such that $a^b = n \in \mathcal{C}^1(\mathcal{N})$. By Lemma 3.8 we know that $d_1(n) = \text{id}_2$ and so $[a] = [n] \in H^1(X, \mathcal{N})$. \square

Lemma 3.14. *Assume X is a 2-simplex and $\mathcal{N} \triangleleft \mathcal{A}$ is a normal subsystem such that, for each $\sigma \subseteq \tau \in \Sigma$ $\alpha_\tau^\sigma: N_\sigma \rightarrow N_\tau$ is an isomorphism. Then $H^1(X, \mathcal{A}) = H^1(X, \mathcal{A}/\mathcal{N})$.*

Proof By Theorem 3.13 it suffices to show that $H^1(X, \mathcal{N}) = 0$ and that κ_1 is onto. Without loss of generality assume that X is the set of non-empty subsets of $\{1, 2, 3\}$. Let $n = (n_{ij}) \in Z^1(X, \mathcal{N})$. That means that $\alpha_{123}^{23}(n_{23}^{-1})\alpha_{123}^{13}(n_{13})\alpha_{123}^{12}(n_{12}^{-1}) = 0$. Because of this, and since all α maps are isomorphisms we can find $m = (m_1, m_2, m_3)$ such that

$$\begin{aligned} \alpha_{23}^3(m_3^{-1})\alpha_{23}^2(m_2) &= n_{23}, \\ \alpha_{13}^3(m_3^{-1})\alpha_{13}^1(m_1) &= n_{13}, \\ \alpha_{12}^2(m_2^{-1})\alpha_{12}^1(m_1) &= n_{12}. \end{aligned}$$

Then, $d_0(m) = \text{id}_1^m = n$. Thus $H^1(X, \mathcal{N}) = 0$. Now take $\bar{a} = (\bar{a}_{ij}) \in Z^1(X, \mathcal{A}/\mathcal{N})$. This means that for a representative $a = (a_{ij})$ we have $d_1(a) = n \in N_{123}$. Now let $a' = (a'_{ij})$ be given by $a'_{12} = a_{12}m^{a_{12}}$, where $n = \alpha_{123}^{12}(m)$ and $a'_{ij} = a_{ij}$ otherwise. Now

$$d_1(a') = \alpha_{123}^{23}(a_{23}^{-1})\alpha_{123}^{13}(a_{13})\alpha_{123}^{12}(a_{12}^{-1})n^{-1} = \text{id}_{123}.$$

Clearly $(\overline{a'_{ij}}) = (\overline{a_{ij}})$ so we are done. \square

3.2. The coefficient system of an amalgam

We now construct a coefficient system $\mathcal{A}_0 = \{A_\bullet, \alpha_\bullet^\bullet\}$, for the amalgam \mathcal{G}_0 by setting for each $\sigma \in \Sigma$

$$A_\sigma = \{g \in \text{Aut}(G_\sigma) \mid g(\overline{G}_{\sigma,\tau}) = \overline{G}_{\sigma,\tau} \text{ for all } \tau \in \Sigma \text{ with } \sigma \subseteq \tau\}$$

(in particular, if σ is maximal, this implies $A_\sigma = \text{Aut}(G_\sigma)$). Moreover, for each pair (σ, τ) with $\sigma \subseteq \tau$ we define $\alpha_\tau^\sigma: A_\sigma \rightarrow A_\tau$ given by $\text{ad}(\psi_\tau^\sigma)$, where $\text{ad}(x)(y) = x^{-1}yx$. If $\phi = \{\phi_\sigma \mid \sigma \in \Sigma\}: \mathcal{G}^{(1)} \rightarrow \mathcal{G}^{(2)}$ is a homomorphism of simplicial amalgams, then $\chi = \{\chi_\sigma = \text{ad}(\phi_\sigma) \mid \sigma \in \Sigma\}: \mathcal{A}_0^{(2)} \rightarrow \mathcal{A}_0^{(1)}$ is a homomorphism of coefficient systems. One verifies easily that this assignment defines a contravariant functor from the category of simplicial amalgams over X to the category of coefficient systems over X .

We now let \mathcal{C}_0^\bullet be the cochain complex associated to \mathcal{A}_0 .

Example 3.15. In Example 2.10 take Γ to be the projective 3-space $\text{PG}(V)$, where V has dimension 4 over \mathbb{F}_2 and let $G = \text{SL}_4(2)$. Thus \mathcal{G}_0 is an amalgam over a complex $X = (V, \Sigma)$, where Σ consists of all non-empty subsets of $V = \{1, 2, 3\}$. A computation with GAP now reveals the following: $\alpha_{12}^1: A_1 \rightarrow A_{12}$ and $\alpha_{23}^3: A_3 \rightarrow A_{23}$ are surjective. This implies that every element in \mathcal{C}_0^1 is in a cohomology class with an element $(\text{id}_{12}, \text{id}_{23}, a_{13})$, for some $a_{13} \in A_{13}$. However, another calculation shows that $\alpha_{123}^{13}: A_{13} \rightarrow A_{123}$ is an isomorphism. Now for $(\text{id}_{12}, \text{id}_{23}, a_{13}) \in Z^1(X, \mathcal{A}_0)$ the 1-cocycle condition forces $a_{13} = \text{id}_{13}$ so that $H^1(X, \mathcal{A}_0) = 0$.

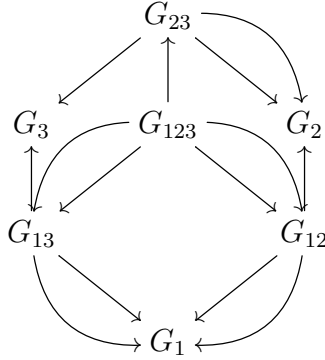
4. The correspondence between cohomology classes and amalgams

Consider the reference amalgam $\mathcal{G}_0 = \{G_\bullet, \psi_\bullet^\bullet\}$ over the connected simplicial complex $X = (V, \Sigma)$. Using Lemma 2.5 we can assume that X has rank at least 2 and contains all 3-subsets of V as a simplex. Recall that every amalgam of type \mathcal{G}_0 has the same target subgroups $\overline{G}_{\sigma,\tau}$. Consider now $\mathcal{G} = \{G_\bullet, \varphi_\bullet^\bullet\}$ a normalized amalgam of type \mathcal{G}_0 . For each $\sigma = \{i, j\} \in \Sigma_1$ with $i < j$ define $a_{ij}^\mathcal{G} = (\varphi_{ij}^i)^{-1} \circ \psi_{ij}^i$. Note that it follows that $a_\sigma \in A_\sigma$ and so the collection $a = \{a_\sigma^\mathcal{G} \mid \sigma \in \Sigma_1\}$ is an element of \mathcal{C}^1 .

Proposition 4.1. *The collection $a = \{a_\sigma^\mathcal{G} \mid \sigma \in \Sigma_2\}$ is an element of $Z^1(X, \mathcal{A}_0)$. Moreover the correspondence $\mathcal{G} \rightarrow \{a_\bullet^\mathcal{G}\}$ is a bijection between the set of normalized amalgams of type \mathcal{G}_0 and the set $Z^1(X, \mathcal{A}_0)$*

Proof We first need to prove that $a \in Z^1(X, \mathcal{A}_0)$. Let us consider $\tau = \{i_1, i_2, i_3\} \in \Sigma_2$, such that $i_1 < i_2 < i_3$. In order to have a cleaner set of notations (and no double subscripts) we will assume that $i_1, i_2, i_3 = 1, 2, 3$. The general case is similar. The amalgam \mathcal{G} is normalized so $\varphi_{123}^{23} = \psi_{123}^{23}, \varphi_{12}^2 = \psi_{12}^2, \varphi_{23}^3 = \psi_{23}^3$ and $\varphi_{13}^3 = \psi_{13}^3$. We also have that $\varphi_{12}^1 = \psi_{12}^1 a_{12}^{-1}, \varphi_{13}^1 = \psi_{13}^1 a_{13}^{-1}$ and $\varphi_{23}^2 = \psi_{23}^2 a_{23}^{-1}$.

The information above is summarised in the following diagram



Here the straight arrows signify the maps ψ_σ^τ and the curved maps signify the maps φ_σ^τ , wherever they differ. Thus, the diagram obtained by considering only straight arrows corresponds to the amalgam \mathcal{G}_0 and the diagram obtained by taking the curved arrows wherever possible corresponds to the amalgam \mathcal{G} . Both these diagrams commute. Note that by the commutativity of the squares, a walk in the diagram for \mathcal{G} from G_{123} to G_{123} along the path

$$\{1, 2, 3\} \rightarrow \{1, 2\} \rightarrow \{1\} \rightarrow \{1, 3\} \rightarrow \{3\} \rightarrow \{2, 3\} \rightarrow \{2\} \rightarrow \{1, 2\} \rightarrow \{1, 2, 3\}$$

gives

$$\text{id}_{G_{123}} = (\psi_{123}^{12})^{-1}(\varphi_{12}^2)^{-1}\varphi_{23}^2(\varphi_{23}^3)^{-1}\varphi_{13}^3(\varphi_{13}^1)^{-1}\varphi_{12}^1\psi_{123}^{12}.$$

Using, that, for $i < j$, we have $\varphi_{ij}^i = \psi_{ij}^i a_{ij}^{-1}$ and $\varphi_{ij}^j = \psi_{ij}^j$, we have

$$\text{id}_{G_{123}} = (\psi_{123}^{12})^{-1}(\psi_{12}^2)^{-1}(\psi_{23}^2 a_{23}^{-1})(\psi_{23}^3)^{-1}(\psi_{13}^3)(a_{13}(\psi_{13}^1)^{-1})(\psi_{12}^1 a_{12}^{-1})\psi_{123}^{12}.$$

Then, using that the diagram commutes we find

$$\begin{aligned} \text{id}_{G_{123}} &= (\psi_{123}^{12})^{-1}\psi_{123}^{12}(\psi_{123}^{23})^{-1}a_{23}^{-1}\psi_{123}^{23}(\psi_{123}^{13})^{-1}a_{13}\psi_{123}^{13}(\psi_{123}^{12})^{-1}a_{12}^{-1}\psi_{123}^{12} \\ &= (\psi_{123}^{23})^{-1}a_{23}^{-1}\psi_{123}^{23}(\psi_{123}^{13})^{-1}a_{13}\psi_{123}^{13}(\psi_{123}^{12})^{-1}a_{12}^{-1}\psi_{123}^{12} \\ &= \alpha_{123}^{23}(a_{23}^{-1})\alpha_{123}^{13}(a_{13})\alpha_{123}^{12}(a_{12}^{-1}). \end{aligned}$$

This last equation means that for any amalgam \overline{G}_\bullet , the collection $a_\bullet^\mathcal{G}$ is a 1-cocycle.

Conversely suppose $a = \{a_\sigma \mid \sigma \in \Sigma_1\}$ is a 1-cocycle. We need to construct an amalgam \mathcal{G} so that $a = \{a_\bullet^\mathcal{G}\}$. We first define φ_σ^τ for any $\sigma \subseteq \tau \in \Sigma$. Let $\max \sigma = j$ and $k = \max \tau$. It follows from Lemma 2.8 that we must define

$$\varphi_\tau^\sigma = \psi_\tau^\sigma \text{ if } j = k.$$

We also define

$$\varphi_\tau^\sigma = (\psi_\sigma^j)^{-1} \circ \psi_{jk}^j a_{jk}^{-1} \circ \psi_\tau^{jk} \text{ if } j < k.$$

Note that this definition is also forced upon us. Namely, for $\sigma = \{j\}$ and $\tau = \{j, k\}$ this is the only possibility since we insist that $a = \{a_\bullet^\mathcal{G}\}$. Moreover, normality already forced us

to set $\varphi_\sigma^j = \psi_\sigma^j$ and $\varphi_\tau^{\{j,k\}} = \psi_\tau^{\{j,k\}}$. Hence our definitions of φ_τ^j and φ_τ^σ are forced upon us by the requirements

$$\varphi_\tau^j = \varphi_{\{j,k\}}^j \circ \varphi_\tau^{\{j,k\}} = \varphi_\sigma^j \circ \varphi_\tau^\sigma.$$

It now suffices to show that for any ρ with $\rho \subseteq \sigma \subseteq \tau$ we have

$$(4.1) \quad \varphi_\tau^\rho = \varphi_\sigma^\rho \circ \varphi_\tau^\sigma.$$

Let $i = \max \rho$ and let j and k be as above. If $i = j = k$ then (4.1) follows since all ψ commute. If $\{i, j, k\} = \{i, k\}$, then either $\varphi_\tau^\sigma = \psi_\tau^\sigma$ or $\varphi_\sigma^\rho = \psi_\sigma^\rho$. Suppose the latter holds. Then,

$$\varphi_\tau^\rho = (\psi_\rho^i)^{-1} \circ \psi_{ik}^i a_{ik}^{-1} \circ \psi_\tau^{ik} = \psi_\sigma^\rho (\psi_\sigma^i)^{-1} \circ \psi_{ik}^i a_{ik}^{-1} \circ \psi_\tau^{ik} = \varphi_\sigma^\rho \circ \varphi_\tau^\sigma$$

and a similar argument holds in the former case. Finally let $i < j < k$. Then, (4.1) amounts to

$$(\psi_\rho^i)^{-1} \circ \psi_{ik}^i a_{ik}^{-1} \circ \psi_\tau^{ik} = (\psi_\rho^i)^{-1} \circ \psi_{ij}^i a_{ij}^{-1} \circ \psi_\sigma^{ij} (\psi_\sigma^j)^{-1} \circ \psi_{jk}^j a_{jk}^{-1} \circ \psi_\tau^{jk}$$

Multiplying by $(\psi_{ijk}^i)^{-1} \psi_\rho^i$ on the left and $(\psi_\tau^{ijk})^{-1}$ on the right, replacing $\psi_\sigma^{ij} (\psi_\sigma^j)^{-1} = (\psi_{ij}^j)^{-1}$, and using that $(\psi_{ij}^j)^{-1} \psi_{jk}^j = \psi_{ijk}^{ij} (\psi_{ijk}^k)^{-1}$ and the ψ all commute this reduces to

$$\alpha_{ijk}^{ik} (a_{ik}^{-1}) = \alpha_{ijk}^{ij} (a_{ij}^{-1}) \alpha_{ijk}^{jk} (a_{jk}^{-1})$$

and this is equivalent to the fact that a is a cocycle. We already demonstrated that, for any 1-cocycle a there is (at most) a unique normalized amalgam \mathcal{G} with $a = \{a_\bullet^\mathcal{G}\}$, so we are done. \square

4.1. Isomorphisms and co-boundaries

Proposition 4.2. *Two normalized amalgams of type \mathcal{G}_0 are isomorphic if and only if the corresponding 1-cocycles are cohomologous.*

Proof For $l = 1, 2$, let ${}^{(l)}\mathcal{G} = \{G_\bullet, {}^{(l)}\varphi_\bullet^\bullet\}$ be a normalized amalgam of type \mathcal{G}_0 corresponding to a cocycle $z^{(l)} = \{z_\sigma^{(l)} \mid \sigma \in \Sigma_1\}$. Recall that this means that, for $\{i, j\} \in \Sigma_1$ with $i < j$ we have

$${}^{(l)}\varphi_{ij}^i = \psi_{ij}^i (z_{ij}^{(l)})^{-1}.$$

Suppose $\phi : \mathcal{G}_\bullet^{(1)} \rightarrow \mathcal{G}_\bullet^{(2)}$ is an isomorphism. It then follows that ${}^{(l)}\varphi_\tau^\tau = \psi_\tau^\tau$ for all τ with $|\tau| > 1$. In particular Equation 2.1 becomes $\phi_\tau^\tau \circ \psi_\tau^\tau = \psi_\tau^\tau \circ \phi_\tau^\tau$ and so we have

$$(4.2) \quad \phi_\tau^\tau = \alpha_\tau^\tau(\phi_\tau^\tau)$$

Moreover if $i < j$, we get the following commutative diagram:

$$(4.3) \quad \begin{array}{ccc} G_i & \xrightarrow{\phi_i} & G_i \\ \psi_{ij}^i \circ (z_{ij}^{(1)})^{-1} = {}^{(1)}\varphi_{ij}^i \uparrow & & \uparrow {}^{(2)}\varphi_{ij}^i = \psi_{ij}^i \circ (z_{ij}^{(2)})^{-1} \\ G_{ij} & \xrightarrow[\phi_{ij} = \alpha_{ij}^j(\phi_j)]{} & G_{ij} \end{array}$$

If we compare the two cocycles $z_{ij}^{(l)} = ({}^{(l)}\varphi_{ij}^i)^{-1} \circ \psi_{ij}^i$ we see that

$$z_{ij}^{(1)} = ({}^{(1)}\varphi_{ij}^i)^{-1} \circ \psi_{ij}^i = \alpha_{ij}^j(\phi_j^{-1}) \circ ({}^{(2)}\varphi_{ij}^i)^{-1} \circ \phi_i \circ \psi_{ij}^i = \alpha_{ij}^j(\phi_j^{-1}) \circ z_{ij}^{(2)} \circ \alpha_{ij}^i(\phi_i).$$

This shows that in fact the diagram (4.3) is commutative if and only if $z^{(1)} = (z^{(2)})^{\{\phi_k | 1 \leq k \leq n\}}$. In particular, $z^{(1)}$ and $z^{(2)}$ are cohomologous. (Note here that since ϕ preserves all G_σ and \overline{G}_σ^τ , we have $\phi_k \in A_k$, for all k .)

Conversely, suppose that $z^{(1)}$ and $z^{(2)}$ are cohomologous, that is, they belong to the same \mathcal{C}^0 -orbit. Let $f = \{f_k \mid k \in V\} \in \mathcal{C}^0$ such that $z^{(1)} = (z^{(2)})^f$. We now define an isomorphism $\phi: {}^{(1)}\mathcal{G} \rightarrow {}^{(2)}\mathcal{G}$. First set

$$\phi_k = f_k \text{ for all } 1 \leq k \leq n.$$

We now note that since ${}^{(1)}\mathcal{G}$ and ${}^{(2)}\mathcal{G}$ are normalized of the same type, Equation (4.2) must be satisfied and inductive use together with the composition properties of the α maps, shows that, for each simplex $\tau = \{i_1, \dots, i_m\}$, with $i_1 < \dots < i_m$, we must have

$$\phi_\tau = \alpha_\tau^{i_m}(f_{i_m}).$$

It now suffices to check that for all simplices $\sigma \subset \tau$ we have

$$\phi_\sigma \circ ({}^{(1)}\varphi_\tau^\sigma = ({}^{(2)}\varphi_\tau^\sigma \circ \phi_\tau.$$

For $\sigma = \{i\}$ and $\tau = \{i, j\}$ with $i < j$, this requires the diagram (4.3) to be commutative and we already saw that this is equivalent to $z^{(1)} = (z^{(2)})^{\{\varphi_k | 1 \leq k \leq n\}}$. Thus for all τ with $|\tau| = 2$, we're done.

Next, consider $\sigma \subset \tau$ where $|\tau| > 2$. Suppose that i and j are the largest vertices of σ and τ respectively. If $i = j$, then

$$\phi_\tau = \alpha_\tau^i(f_i) = \alpha_\tau^\sigma \circ \alpha_\sigma^i(f_i) = \alpha_\tau^\sigma(\phi_\sigma) = (\psi_\tau^\sigma)^{-1} \circ \phi_\sigma \circ \psi_\tau^\sigma = ({}^{(2)}\varphi_\tau^\sigma)^{-1} \circ \phi_\sigma \circ ({}^{(1)}\varphi_\tau^\sigma$$

and we are done. Here the last equality follows from Lemma 2.8.

If $i \neq j$, then $i < j$. Since, for $l = 1, 2$, ${}^{(l)}\mathcal{G}$ is normalized, we have ${}^{(l)}\varphi_\sigma^i = \psi_\sigma^i$ and ${}^{(l)}\varphi_\tau^{ij} = \psi_\tau^{ij}$, again by Lemma 2.8. It follows that the left and right square in the diagram below are commutative.

$$\begin{array}{ccccc}
 G_\sigma & \xleftarrow{({}^{(2)}\varphi_\sigma^i = \psi_\sigma^i)} & G_i & \xleftarrow{({}^{(2)}\varphi_{ij}^i)} & G_{ij} & \xleftarrow{({}^{(2)}\varphi_\tau^{ij} = \psi_\tau^{ij})} & G_\tau \\
 & \searrow & \uparrow f_i = \phi_i & \searrow & \uparrow \phi_{ij} = \alpha_{ij}^j(f_j) & \searrow & \uparrow \phi_\tau = \alpha_\tau^j(f_j) \\
 & & G_i & \xleftarrow{({}^{(1)}\varphi_{ij}^i)} & G_{ij} & & \\
 & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \\
 G_\sigma & \xleftarrow{({}^{(1)}\varphi_\sigma^i = \psi_\sigma^i)} & G_i & \xleftarrow{({}^{(1)}\varphi_\tau^\sigma)} & G_{ij} & \xleftarrow{({}^{(1)}\varphi_\tau^{ij} = \psi_\tau^{ij})} & G_\tau
 \end{array}$$

Moreover, the middle square is commutative because of the rank-2 case. The top and bottom square are also commutative and so the result follows. \square

The proof of Theorem 1 is now complete.

Example 4.3. In Example 3.15 we considered the special case of Example 2.10, where the amalgam \mathcal{G}_0 consists of standard parabolic subgroups of $\mathrm{SL}_4(2)$ as it acts on the projective 3-space $\mathrm{PG}(V)$, where $V = \mathbb{F}_2^4$. We computed that $H^1(X, \mathcal{A}) = 0$, so that by Theorem 1 there exists a unique amalgam of this type.

5. Amalgams over small complexes

5.1. Goldschmidt's Lemma

The simplest case of the theory is the celebrated Goldschmidt's Lemma. It arises as follows from our setup. Let X be the 1-simplex $\{\{1\}, \{2\}, \{1, 2\}\}$ and let \mathcal{G}_0 be a generic amalgam over X . We denote its three groups by G_1, G_2, G_{12} and let $\psi^i: G_{12} \rightarrow G_i$, for $i = 1, 2$. We also define the three automorphisms groups as in Subsection 3.2 as $A_{12} = \mathrm{Aut}(G_{12})$ and

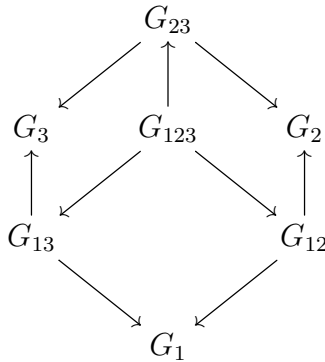
$$A_i = \{g \in \mathrm{Aut}(G_i) \mid g(\overline{G_{i,\{1,2\}}}) = \overline{G_{i,\{1,2\}}}\}$$

respectively $\overline{A}_i = \mathrm{ad}(\psi^i)(A_i) \leq A_{12}$. Theorem 1 now reads as

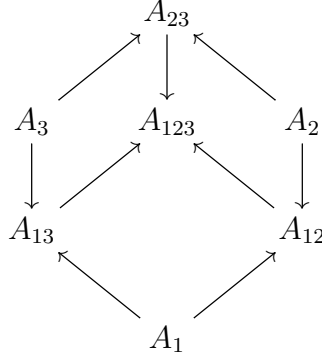
Corollary 5.1. (Goldschmidt's Lemma, see [13, §2.7] and [11, Ch. 16]) *There is a 1-1 correspondence between isomorphism classes of amalgams of type $\mathcal{A}_0 = \{G_1, G_2, G_{12}\}$ and double cosets of \overline{A}_1 , and \overline{A}_2 in A_{12} .*

5.2. Triangular complexes

Let X be the 2-dimensional simplex consisting of all non-empty subsets of $V = \{1, 2, 3\}$ and let \mathcal{G}_0 be a generic amalgam over X . Such an amalgam arises naturally from a group acting flag-transitively on a rank 3 geometry as in Example 2.10.



In order to apply Theorem 1, we need to consider the corresponding coefficient system \mathcal{A} .



Here $A_i = \{g \in \text{Aut}(G_i) \mid g(\overline{G}_{ij}) = \overline{G}_{ij} \text{ for all } j \neq i\}$, $A_{ij} = \{g \in \text{Aut}(G_{ij}) \mid g(\overline{G}_{123}) = \overline{G}_{123}\}$, $A_{123} = \text{Aut}(G_{123})$ and the arrows represent the maps $\alpha_\tau^\sigma: A_\sigma \rightarrow A_\tau$, which can be viewed as restriction maps.

We shall consider $N = \text{Inn}(G_{123})$ and note that for any σ there is a group $N_\sigma \triangleleft A_\sigma$ so that $N \cong N_\sigma$ and that α_τ^σ restricted to N_σ gives an isomorphism between N_σ and N_τ . We define then $\overline{A}_\sigma = A_\sigma/N_\sigma$ and $\overline{\alpha}_\tau^\sigma: \overline{A}_\sigma \rightarrow \overline{A}_\tau$. Define then the set

$$\overline{\mathcal{C}}^1 = \{(a_{23}, a_{13}, a_{12}) \mid a_{ij} \in \overline{A}_{ij} \text{ and } \overline{\alpha}_{123}^{23}(a_{23}^{-1})\overline{\alpha}_{123}^{13}(a_{13})\overline{\alpha}_{123}^{12}(a_{12}^{-1}) = id_{\overline{A}_{123}}\}$$

The group $\overline{\mathcal{C}}^0 = \overline{A}_1 \times \overline{A}_2 \times \overline{A}_3$ acts on $\overline{\mathcal{C}}^1$ via

$$(a_{23}, a_{13}, a_{12})^{(d_1, d_2, d_3)} = (\overline{\alpha}_{23}^3(d_3^{-1})a_{23}\overline{\alpha}_{23}^2(d_2), \overline{\alpha}_{13}^3(d_3^{-1})a_{13}\overline{\alpha}_{23}^1(d_1), \overline{\alpha}_{12}^2(d_2^{-1})a_{12}\overline{\alpha}_{12}^1(d_1))$$

Proposition 5.2. *The isomorphism classes of amalgams of type \mathcal{G}_0 are in bijection to the orbits of $\overline{\mathcal{C}}^0$ on $\overline{\mathcal{C}}^1$*

Proof Immediate from Theorem 1 and Lemma 3.14 □

Remark 5.3. Note that the group and the action is exactly as in the example of triangular rank two amalgams from [7], however the existence of the group G_{123} gives smaller A 's and insures that we only need to take the orbits in $\overline{\mathcal{C}}^1$.

6. Generalizations

The techniques we have developed in this paper so far can be used in the more general setting of concrete categories. In this section we merely outline this generalization since the translations are rather straightforward and we are mainly interested in group theory. Our main reference for (concrete) categories is [1].

We start by recalling a few definitions from category theory. For a category \mathbf{C} , we denote the collection of objects $\text{obj}(\mathbf{C})$ and for $A, B \in \text{obj}(\mathbf{C})$ we let $\text{hom}_{\mathbf{C}}(A, B)$ denote the collection of \mathbf{C} -morphisms from A to B

Definition 6.1. A *concrete category* (over **Set**) is a category **C** equipped with a functor $F : \mathbf{C} \rightarrow \mathbf{Set}$ that is faithful, that is, for objects $A, B \in \text{obj}(\mathbf{C})$, the hom-set restrictions

$$F : \text{hom}_{\mathbf{C}}(A, B) \rightarrow \text{hom}_{\mathbf{Set}}(FA, FB)$$

are injective. The functor F is called the *forgetful functor*. In [1] a concrete category over **Set** is called a *construct*.

Since a given category might be concrete over several categories, the functor F is part of the definition of a concrete category. For each $A \in \text{obj}(\mathbf{C})$ we shall write $|A| = F(A)$. Moreover, using that F is faithful, we shall abusively identify each $f \in \text{hom}_{\mathbf{C}}(A, B)$ with the set function $F(f) \in \text{hom}_{\mathbf{Set}}(|A|, |B|)$. Note however, that in general not all functions $f : |A| \rightarrow |B|$ correspond to morphisms of **C**.

Definition 6.2. A morphism $A \xrightarrow{f} B$ is called an *embedding* if

- (1) the underlying function $|A| \xrightarrow{f} |B|$ is injective, and
- (2) f is initial, i.e. for any $C \in \text{obj}(\mathbf{C})$, a function $|C| \xrightarrow{g} |A|$ underlies a **C**-morphism whenever $|C| \xrightarrow{f \circ g} |B|$ underlies a **C**-morphism.

Remark 6.3. Note that condition 2. ensures that if you can compose a morphism with f^{-1} in **Set** then you can do so in **C**. More precisely, suppose that $h \in \text{hom}_{\mathbf{C}}(C, B)$ is so that $h(|C|) \subseteq f(|A|)$. We can then define the composition of functions $g = f^{-1} \circ h \in \text{hom}_{\mathbf{Set}}(|C|, |A|)$. Now since $f \circ g = h$ and h is a **C** morphism, so is g .

We now consider a simplicial complex $X = (V, \Sigma)$ and a concrete category (\mathbf{C}, F) . The simplicial complex X gives rise to its poset category **X** in which $\text{obj}(\mathbf{X}) = \Sigma$ and

$$\text{hom}_{\mathbf{X}}(A, B) = \begin{cases} \{ "A \leq B" \} & \text{if } A \leq B \\ \emptyset & \text{if } A \not\leq B \end{cases}$$

In this setting, an amalgam (cf. Definition 2.2) is a functor.

Definition 6.4. A *simplicial amalgam over X in the (concrete) category C* is a contravariant functor $\mathcal{G} : \mathbf{X} \rightarrow \mathbf{C}$ such that for any pair of simplices σ, τ with $\sigma \leq \tau$, the map $\mathcal{G}(" \sigma \leq \tau ") : \mathcal{G}(\tau) \rightarrow \mathcal{G}(\sigma)$ is an embedding (so in particular $\mathcal{G}(" \sigma \leq \tau ")$ is a monomorphism). By analogy with the group theoretic setting we shall denote $\mathcal{G}(\sigma)$ by \mathcal{G}_σ and $\mathcal{G}(" \sigma \leq \tau ")$ by φ_τ^σ and write $\mathcal{G} = (\mathcal{G}_\bullet, \varphi_\bullet)$.

Since **X** and **C** are fixed in this section, we shall simply call \mathcal{G} an amalgam.

Definition 6.5. For an amalgam \mathcal{G} and each $\sigma, \tau \in \Sigma$ with $\sigma \leq \tau$ we define

$$|\overline{\mathcal{G}}_{\sigma, \tau}| = \varphi_\tau^\sigma(|\mathcal{G}_\tau|) \subseteq |\mathcal{G}_\sigma|.$$

Note that this is a set, not an object of **C**. Given an amalgam ${}^0\mathcal{G}$, an *amalgam of type* ${}^0\mathcal{G}$ is an amalgam \mathcal{G} such that $\mathcal{G}_\sigma = {}^0\mathcal{G}_\sigma$ for all $\sigma \in \Sigma$ and $|\overline{{}^0\mathcal{G}}_{\sigma, \tau}| = |\overline{\mathcal{G}}_{\sigma, \tau}|$ for all $\sigma, \tau \in \Sigma$ with $\sigma \leq \tau$. We call \mathcal{G} *normalized* if for any simplex τ we have $\mathcal{G}(" \bar{\tau} \leq \tau ") = {}^0\mathcal{G}(" \bar{\tau} \leq \tau ")$.

In this setting, a morphism of amalgams (see Definition 2.3) is a natural transformation.

Definition 6.6. We define a *homomorphism* between the amalgams $\mathcal{G}^{(1)} = \{\mathcal{G}_\bullet^{(1)}, {}^{(1)}\varphi_\bullet\}$ and $\mathcal{G}^{(2)} = \{\mathcal{G}_\bullet^{(2)}, {}^{(2)}\varphi_\bullet\}$ to be a natural transformation $\phi: \mathcal{G}^{(1)} \rightarrow \mathcal{G}^{(2)}$. That is, $\phi = \{\phi_\sigma \mid \sigma \in \Sigma\}$ where $\phi_\sigma \in \text{hom}_{\mathbf{C}}(\mathcal{G}_\sigma^{(1)}, \mathcal{G}_\sigma^{(2)})$ is such that, when $\sigma \leq \tau$ we have

$$(6.1) \quad \begin{array}{ccc} \mathcal{G}_\sigma^{(1)} & \xrightarrow{\phi_\sigma} & \mathcal{G}_\sigma^{(2)} \\ {}^{(1)}\varphi_\tau^\sigma \uparrow & & \uparrow {}^{(2)}\varphi_\tau^\sigma \\ \mathcal{G}_\tau^{(1)} & \xrightarrow{\phi_\tau} & \mathcal{G}_\tau^{(2)} \end{array}$$

We call ϕ an *isomorphism* of amalgams if it is a natural isomorphism, that is, if ϕ_σ is an isomorphism for all $\sigma \in \Sigma$.

Definition 6.7. The simplicial amalgams over \mathbf{X} in \mathbf{C} and homomorphisms of such amalgams form the objects and morphisms of a category that we shall denote $\mathbf{Amal}_{\mathbf{X}}(\mathbf{C})$. It is a subcategory of the functor quasi-category $[\mathbf{X}, \mathbf{C}]$, which itself is (isomorphic to) a category since \mathbf{X} is a small category (see [1]).

One can now easily reprove Proposition 2.9. Next, we define what is the coefficient system associated to an amalgam $\mathcal{G} = \{\mathcal{G}_\bullet, \psi_\bullet\}$. Recall that an automorphism of $A \in \text{obj}(\mathbf{C})$ is an isomorphism in $\text{hom}_{\mathbf{C}}(A, A)$. The collection of all automorphisms of A is denoted $\text{Aut}_{\mathbf{C}}(A)$ and forms a group.

Definition 6.8. The coefficient system associated to the amalgam $\mathcal{G} = \{\mathcal{G}_\bullet, \psi_\bullet\}$ is the (covariant) functor $\mathcal{A}: \mathbf{X} \rightarrow \mathbf{Group}$, where for each simplex $\sigma \in \Sigma = \text{obj}(\mathbf{X})$ we have

$$\mathcal{A}(\sigma) = \{g \in \text{Aut}_{\mathbf{C}}(\mathcal{G}_\sigma) \mid g(|\overline{\mathcal{G}}_{\sigma,\tau}|) = |\overline{\mathcal{G}}_{\sigma,\tau}| \text{ for all } \tau \text{ with } \sigma \leq \tau\},$$

and for every $\sigma, \tau \in \Sigma$ with $\sigma \leq \tau$ we have

$$\mathcal{A}(\sigma \leq \tau): A_\sigma \rightarrow A_\tau \\ f \mapsto (\psi_\tau^\sigma)^{-1} \circ f \circ \psi_\tau^\sigma.$$

As before we shall write $\mathcal{A} = \{A_\bullet, \alpha_\bullet\}$, where $A_\sigma = \mathcal{A}(\sigma)$ and $\alpha_\tau^\sigma = \mathcal{A}(\sigma \leq \tau)$.

Remark 6.9. Since morphisms of amalgams are required to be embeddings, it follows from Remark 6.3 that the α 's are well-defined. Moreover, since $((\psi_\tau^\sigma)^{-1} \circ f \circ \psi_\tau^\sigma)^{-1} = (\psi_\tau^\sigma)^{-1} \circ f^{-1} \circ \psi_\tau^\sigma$, α_τ^σ maps A_σ to A_τ .

It is now immediate to prove Propositions 4.1 and 4.2 in this categorical setting and Theorem 2 follows.

Of course one can apply these generalizations to many natural categories such as **Ring**, **Top** and so on. We choose to exemplify with another category of group theoretic flavour. We refer the reader to [20] for details.

6.1. Sheaves on geometries.

We shall define the notion of a sheaf on a geometry in the sense of Ronan and Smith (cf. [20, 21]) and show that it is an example of an amalgam over a simplicial complex.

Let k be a field and let G be a finite group. Let Γ be an incidence geometry with type set I on which G acts flag transitively. Let $X(\Gamma) = (V(\Gamma), \Sigma(\Gamma))$ be the simplicial complex whose vertices are the objects of Γ and whose simplices are the flags of Γ . We let G act on $X(\Gamma)$ in accordance with its action on Γ itself.

Definition 6.10. A (G -equivariant) *sheaf* on $X(\Gamma)$ is a collection

$$\mathcal{F}_\Gamma = \{\mathcal{F}_\sigma, \phi_\tau^\sigma \mid \sigma \subseteq \tau \in \Sigma(\Gamma)\},$$

where \mathcal{F}_σ is a k -vector space and $\phi_\tau^\sigma: \mathcal{F}_\tau \rightarrow \mathcal{F}_\sigma$ is a linear map whenever $\sigma \subseteq \tau \in \Sigma$. Moreover, $\phi_\rho^\sigma \circ \phi_\tau^\rho = \phi_\tau^\sigma$ whenever $\sigma \leq \rho \leq \tau$. Finally we require that G acts on the set $\prod_\sigma \mathcal{F}_\sigma$. More precisely, $g \in G$ acts linearly via $\tilde{g}_\sigma: \mathcal{F}_\sigma \rightarrow \mathcal{F}_{\sigma g}$ and the following diagram commutes.

$$(6.2) \quad \begin{array}{ccc} \mathcal{F}_\sigma & \xrightarrow{\tilde{g}_\sigma} & \mathcal{F}_{\sigma g} \\ \phi_\tau^\sigma \uparrow & & \uparrow \phi_{\tau g}^{\sigma g} \\ \mathcal{F}_\tau & \xrightarrow{\tilde{g}_\tau} & \mathcal{F}_{\tau g} \end{array}.$$

Suppose moreover that G is transitive on the chambers of the geometry Γ . In this case, much of the information contained in a sheaf is redundant. For more details on the construction see Section 2 of [21]. Indeed, let us take c a chamber of the geometry and for each subflag σ of c consider G_σ , the stabilizer of σ . A *stalk at c* is a system

$$\mathcal{F} = \{\mathcal{F}_\sigma, \phi_\tau^\sigma \mid \sigma \subseteq \tau \subseteq c\},$$

where \mathcal{F}_σ is a kG_σ -module and ϕ_τ^σ is a kG_σ -module homomorphism whenever $\sigma \subseteq \tau \subseteq c$.

We then have (see theorem 2.3 of [21])

Theorem 6.11. *Every stalk is the restriction (to the faces of c) of a unique sheaf.*

We shall now consider the category **GRep** of pairs (G, V) where G is a (finite) group and V is a k representation of G . A homomorphism in $\text{hom}_{\mathbf{GRep}}((G_1, V_1), (G_2, V_2))$ is a pair (ϕ, ψ) , where $\phi: G_1 \rightarrow G_2$ is a group homomorphism, $\psi: V_1 \rightarrow V_2$ is a linear map, and for all $g \in G_1$ and $v \in V_1$, we have

$$\psi(gv) = \phi(g)\psi(v).$$

The functor $F: \mathbf{GRep} \rightarrow \mathbf{Set}$, $F((G, V)) = G \times V$ and $F((\phi, \psi)) = \phi \times \psi$ makes (\mathbf{GRep}, F) into a concrete category. Let X be the simplex of faces of c . We then can define $\mathbf{Amal}_X(\mathbf{GRep})$ to be the category of simplicial amalgams over X in **GRep**. There is an obvious forgetful functor $\mathcal{G}: \mathbf{Amal}_X(\mathbf{GRep}) \rightarrow \mathbf{Amal}_X(\mathbf{Group})$ from this category into the category of simplicial amalgams over X in the category **Group**. The following is now quite clear.

Theorem 6.12. *The category of sheaves over Γ is equivalent to the fiber category $\mathcal{G}_\mathcal{A}$ where \mathcal{A} is the amalgam of parabolics of G defined by Γ .*

This suggests that the category $\mathbf{Amal}_X(\mathbf{GRep})$ would be interesting to study as the various fiber categories will say something about the representation theory of various groups.

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